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Periodic graphs and connectivity of the rational digital hyperplanes

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Abstract

Given a *digital hyperplane* of $\mathbb{Z}^n$ defined by a double-inequality $h \leqslant \sum_{i=1}^{n} a_i x_i < h + \delta$, we want to determine whether it is *connected*. The problem consists of computing the connectivity of a graph whose set of vertices is not finite. The classical algorithms of labelling are not deterministic in this framework but we can think of using the properties of the digital hyperplanes and in particular their periodicity to provide a deterministic method. It leads to introduce a special kind of graphs that we call *periodic* and whose properties allow to compute connective components of infinite size. It provides a deterministic algorithm determining whether a given rational digital hyperplane is connected. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Discrete connectivity; Digital hyperplane; Periodic graph.

1. Introduction

We consider any digital hyperplane $H \subset \mathbb{Z}^n$ defined by a double inequality $h \leqslant \sum_{i=1}^{n} a_i x_i < h + \delta$ and the problem is to know whether it is connected or not. Many discrete connectivities can be considered but in order to stay in the largest framework as possible, we will use a practical and generic definition of this kind of notions. When the dimension n is 2, it is known since 1991 [8] that the question has an arithmetic solution: by denoting m the number of multiples of $\gcd(a_1, a_2)$ belonging to the interval $[h, h+\delta[$, the digital hyperplane H of $\mathbb{Z}^2$ (in fact a digital straight line) is 8-connective if and only if $m \geqslant \max(|a_1|, |a_2|)$ and it is 4-connective if and only if $m \geqslant |a_1| + |a_2|$. When the dimension n becomes greater than 2, the difficulty of the problem increases and no arithmetic solution has been found, even for $n = 3$. The question has been mentioned many times in DGCI conferences but no arithmetic result has allowed to close the question. It is the reason why we choose here another approach, which is still a little arithmetic but mainly algorithmic. Given $(a_i)_{1 \leqslant i \leqslant n} \in \mathbb{R}^n/(0)_{1 \leqslant i \leqslant n}$, $h \in \mathbb{R}$ and $\delta \in \mathbb{R}^+$,

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the digital hyperplane H of double inequality $h \leqslant \sum_{i=1}^{n} a_i x_i < h + \delta$ is a computable subset of $\mathbb{Z}^n$ which is said to be connected if and only if the graph of adjacency of its points is connected. Then the question is the connectivity of a graph but the set of its vertices is not finite. In the rational case, the graphs of connectivities of the digital hyperplanes have some invariance properties which make them special and which lead to introduce a new kind of graph that we call *periodic*. In a relatively general case, the question of their connectivity is decidable and as the graphs of connectivity of the rational digital hyperplanes are of this kind, it provides an algorithm computing the connectivity of the rational digital hyperplanes.

2. Digital hyperplanes

2.1. Definitions

The notion of digital hyperplane was introduced by Reveillès at the beginning of the nineties [8].

Definition 2.1. A *digital hyperplane* of $\mathbb{Z}^n$ is a non empty subset of points $x \in \mathbb{Z}^n$ which can be characterized by a double inequality of the form

$$h \leqslant \varphi(x) < h + \delta,$$

where $h \in \mathbb{R}$, where $\delta \in \mathbb{R}^+$ and where φ denotes a linear form of $\mathbb{R}^n$.

Remark. In continues geometry, an hyperplane can be characterized by many different equations because it is sufficient to multiply an equation by a coefficient different from zero to obtain a new one. In digital geometry, it is the same: many different double inequalities characterize the same digital hyperplane.

The Definition 2.1 is a bit more general than the traditional one where φ is supposed to be only a linear form of $\mathbb{Q}^n$. This leads us to distinguish two different kinds of digital hyperplanes:

- the traditional digital hyperplanes, namely the digital hyperplanes that one can characterize by a double inequality whose linear form is in $\mathbb{Q}^n$ (the boundaries do not matter). We call them *rational*,
- and the digital hyperplanes which cannot be characterized by a linear form of $\mathbb{Q}^n$. We call them *irrational*.

Given a double inequality, it is easy to determine whether a digital hyperplane is rational:

Theorem 2.1. *The digital hyperplane H characterized by the double inequality*

$$h \leqslant \sum_{i=1}^{n} a_i x_i < h + \delta,$$

where $(a_i)_{1\leqslant i\leqslant n}\in\mathbb{R}^n/\{(0)_{1\leqslant i\leqslant n}\}$ *and where* a_j *is supposed to be different from* 0, *is rational if and only if* $(a_i/a_j)_{1\leqslant i\leqslant n}\in\mathbb{Q}^n$.

Proof. This theorem is a direct consequence of the unicity of the normal direction of a given digital hyperplane or in other words, to the fact that two linear forms giving double inequations of the same digital hyperplane are necessarily colinear [5]. □

The rational digital hyperplanes and the irrational ones do not exactly have the same properties and the main difference is their periodicity.

2.2. Periodicity

The periodicity that we use here is defined from the action of the translations.

Definition 2.2. Let Q be a part of the additive abelian group $\mathbb{Z}^n$.

The set of the translations of $\mathbb{Z}^n$ which let Q be invariant, identifies with a subgroup of $\mathbb{Z}^n$ called the *stabilizer* of Q and denoted as G_Q.

We consider an equivalence relation on Q: two points of Q are *equivalent* if their difference belongs to the stabilizer of Q. The equivalence classes of this relation are called the *orbits* of Q.

The set of the orbits is called the *period* of Q and is denoted P_Q.

Remark. The orbits of Q make a partition of the set Q.

Example. Let Q be the subset $\{(x,y)\in\mathbb{Z}^2/x=1,\ x=2 \text{ or } x=3\}$ of $\mathbb{Z}^2$. Its stabilizer is $\mathbb{Z}.(0,1)$ and its period contains three orbits, the one of $(1,0)$, the one of $(2,0)$ and the one of $(3,0)$.

2.3. Periodicity of the digital hyperplanes

One of the main property of the rational digital hyperplanes is their periodicity.

Theorem 2.2. *A digital hyperplane has a finite period if and only if it is rational.*

Proof. Let H be any digital hyperplane of double inequality $h\leqslant\varphi(x)<h+\delta$. Its stabilizer is $G_H=\{y\in\mathbb{Z}^n/\varphi(y)=0\}$ and then there is one orbit for each value of $\varphi(\mathbb{Z}^n)$ belonging to $[h,h+\delta[$. We end the proof by seeing that the set $\varphi(\mathbb{Z}^n)$ has an infinite number of values in $[h,h+\delta[$ if and only if there exists no real λ such that $\lambda.\varphi$ is a linear form of $\mathbb{Q}^n$. □

The finite period of the digital rational hyperplanes is the key point of the deterministic computation of their connective components.

3. Discrete connectivities

3.1. Definitions

We begin by recalling the usual definitions of graph and of connectivity.

Definition 3.3. A *Graph* Γ is a pair (V,E) where V is a set whose elements are called *vertices* and where E is a symmetric subset of $V\times V$ whose elements are called *edges*.

Definition 3.4. *Let* $\Gamma=(V,E)$ be a graph.

A *path* of Γ is a finite sequence of vertices $v_i\in V$ (for $0\leqslant i\leqslant l$) verifying $(v_i,v_{i+1})\in E$ (for any $i\in[|0,l-1|]$).

If there exists a path whose first element v_0 is x and whose last one is y, we say that x and y are *connected*. It is an equivalence relation whose classes are called *connective components*. The class of x is denoted $component_\Gamma(x)$.

The graph Γ is *connected* if there exists only one connective component.

In the seventies, Rosenfeld has introduced several notions of discrete connectivities related with graph structures [6]. We can define them as others by using a general notion of discrete neighbourhood.

Definition 3.5. We call *neighbourhood of the origin* a symetric finite subset N of $\mathbb{Z}^n$ (if $x\in N$ then $-x\in N$).

Let Q be a part of $\mathbb{Z}^n$. We define the *graph of N-adjacency* of Q by its vertices and its edges:

- its vertices are the points of Q,
- two vertices $a\in Q$ and $b\in Q$ are joined by an edge if and only if their difference $b-a$ belongs to N.

We use this graph structure to define the notion of N-connectivity.

Definition 3.6. Let N be a neighbourhood of the origin. The subset $Q\subset\mathbb{Z}^n$ is *N-connected* if and only if its graph of N-adjacency is connected.

We have choosen this definition rather than another because it is short and general. The most useful neighbourhoods of the origin are the balls of radius 1 around the origin for the distances $d_1((x_i)_{1\leqslant i\leqslant n},(y_i)_{1\leqslant i\leqslant n})=\sum_{i=1}^n|y_i-x_i|$ and $d_\infty((x_i)_{1\leqslant i\leqslant n},(y_i)_{1\leqslant i\leqslant n})=\max\{|y_i-x_i|/1\leqslant i\leqslant n\}$ namely

$$N_1=\left\{(x_i)_{1\leqslant i\leqslant n}\in\mathbb{Z}^n\Big/\sum_{i=1}^n|x_i|\leqslant 1\right\},$$

and

$$N_\infty = \{(x_i)_{1 \leqslant i \leqslant n} \in \mathbb{Z}^n, \max\{|x_i|/1 \leqslant i \leqslant n\} \leqslant 1\}.$$

When dimension n is 2, the N_1-connectivity is exactly the classical 4-connectivity and the N_∞-connectivity is the 8-connectivity. When dimension n is 3, the N_1-connectivity is the 6-connectivity and the N_∞-connectivity is the 26-connectivity.

3.2. Decidability of the N-connectivity of the digital hyperplanes?

Let N be a neighbourhood of the origin in $\mathbb{Z}^n$ and H be the digital hyperplane characterized by the double inequality

$$h \leqslant \sum_{i=1}^{n} a_i x_i < h + \delta,$$

where $h \in \mathbb{R}$, $\delta \in \mathbb{R}$ and where $(a_i)_{1 \leqslant i \leqslant n} \in \mathbb{R}^n / \{(0)_{1 \leqslant i \leqslant n}\}$. The question is to determine whether H is N-connected. By definition, it is equivalent to determine whether the graph of N-adjacency of H is connected. This graph has an infinite number of vertices and it makes the classical labelling algorithms non-deterministic. Then the decidability of the N-connectivity of H is not trivial and if we want to determine in a finite time whether H is N-connected, we need to use the properties of its graph of N-adjacency.

3.3. Periodicity of the graphs of N-adjacency

Let Q be any part of $\mathbb{Z}^n$, G_Q its stabilizer and P_Q its period. By assuming that we have choosen a representative of each orbit, we can denote *orbit*(a) the orbit of any element $a \in Q$ and *representative*(a) the element that we have chosen to represent it. With these notations, the map $\alpha : Q \to P_Q \times G_Q$ defined by $\alpha(a) = (orbit(a), a - representative(a))$ is one to one and by using this identification, we can see the graph of N-adjacency of Q as a graph on the set of vertices $P_Q \times G_Q$.

Property 3.1. *Let (x, g) and (x', g') be two elements of $P_Q \times G_Q$ and let k be an element of the stabilizer G_Q. The graph of N-adjancency of Q (seen as a graph on $P_Q \times G_Q$) has an edge between (x, g) and (x', g') if and only if it has an edge between $(x, g+k)$ and $(x', g'+k)$.*

Proof. By definition, the differences $\alpha^{-1}(x', g') - \alpha^{-1}(x, g)$ and $\alpha^{-1}(x', g'+k) - \alpha^{-1}(x, g+k)$ are equal and it follows that one belongs to N if and only if the other also belongs to it. $\square$

The graph of N-adjacency of Q seen as a graph on $P_Q \times G_Q$ has a particular structure that we call *periodic* and that we are going to detail now.

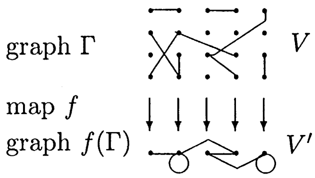

Fig. 1. The image of the graph Γ by the map f.

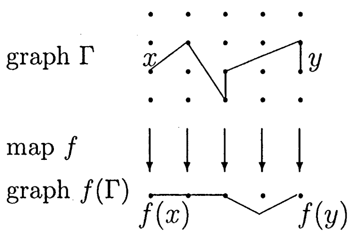

Fig. 2. A path joining two vertices of Γ and its image by f.

4. Periodic graphs

4.1. Preliminaries

We have to define the action on a graph $\Gamma=(V,E)$ of a map acting on the set of the vertices V.

Definition 4.7. The *image graph* of $\Gamma=(V,E)$ by the map $f:V\to V'$ is the graph $f(\Gamma)=(V',(f\otimes f)(E))$ where $f\otimes f:V\times V\to V'\times V'$ is defined by $(f\otimes f)(v,v')=(f(v),f(v'))$ (Fig. 1).

Property 4.2. *The image of the connective component* $\text{component}_\Gamma(x)$ *by the map* f *is included in the connective component of the vertex* $f(x)$ *in the graph* $f(\Gamma)$:

$$f(\text{component}_\Gamma(x))\subset \text{component}_{f(\Gamma)}(f(x)).$$

Proof. The image by f of a path joining two vertices x and y in the graph Γ is a path joining $f(x)$ and $f(y)$ in the image graph $f(\Gamma)$ (Fig. 2). $\square$

Remark. The converse inclusion is false in general because a path in the graph $f(\Gamma)$ cannot always be raised up in a path of Γ (Fig. 3).

4.2. Definition

A periodic graph is a graph defined on a set of vertices $V=P\times G$ where P is any set, G a group, and whose set of edges verifies a property of invariance by *translations*:

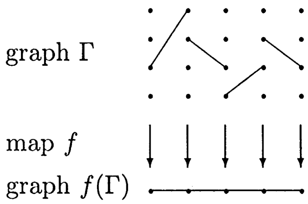

Fig. 3. A path of $f(\Gamma)$ which cannot be raised up in a path of Γ.

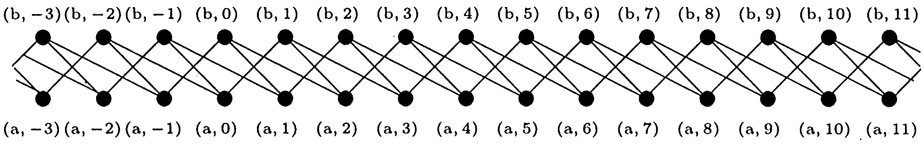

Fig. 4. A part of a periodic graph of period $\{a, b\}$ and of group $\mathbb{Z}$.

Vocabulary. For any element $k \in G$, the map

$$translation_k : V \to V$$
$$translation_k(x, g) = (x, g + k)$$

is called a *translation* of $V = P \times G$.

Definition 4.8. A graph Γ defined on the set of vertices $V = P \times G$ is *periodic* if it is invariant by any translation:

$$\forall k \in G, \ translation_k(\Gamma) = \Gamma.$$

Remarks. (1) For any element $k \in G$, the set E of the edges of a periodic graph $\Gamma = (P \times G, E)$ verifies $((x, g), (y, h)) \in E$ if and only if $((x, g + k), (y, h + k)) \in E$.

(2) The periodicity is here defined for the right translations and it is clear that a notion of "left" periodicity can be obtained by using the left translations.

(3) The graph of N-adjacency of any subset Q of $\mathbb{Z}^n$ is periodic when it is seen as a graph on the set of vertices $P_Q \times G_Q$ (Fig. 4).

Vocabulary. The set P is called the *period* of the graph and G its *group*.

Remark. Any graph $\Gamma = (V, E)$ can be seen as a periodic graph with the set G of all the bijections preserving Γ as group and the set of the orbits of V according to G as period (Fig. 5).

4.3. Properties

The properties that we are going to investigate concern the action of a morphism and its effects on the connective components.

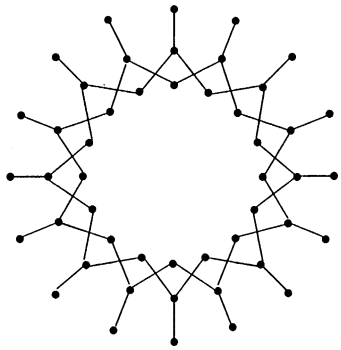

Fig. 5. A periodic graph with a period of three elements and of group $\mathbb{Z}/16\mathbb{Z}$.

4.3.1. Properties of the connective components of a periodic graph

Let Γ be a periodic graph of period P and of group G. The following lemma is a direct consequence of the periodicity.

Lemma 4.1. *For any element k of the group G, there exists a path in Γ joining the vertices (x,g) and (y,h) if and only if there exists one joining $(x,g+k)$ and $(y,h+k)$.*

Property 4.3. *A connective component* $\mathrm{component}_\Gamma(x,g)$ *of the periodic graph Γ is invariant by the translation* $\mathrm{translation}_k$ *(where $k\in G$) if and only if it contains two elements of the form (y,h) and $(y,h+k)$.*

Proof. If the connective component $component_\Gamma(x,g)$ is invariant by translation of element k, then we can simply choose $y=x$ and $h=g$.

In order to show the converse, we assume that two elements (y,h) and $(y,h+k)$ both belong to the connective component of (x,g) and we are going to prove that if (z,l) belongs to this connective component then it is also the case of $(z,l+k)$. If (z,l) belongs to the connective component of (x,g), then there exists a path lying (y,h) and (z,l). According to Lemma 4.1, there exists a path joining $(y,h+k)$ and $(z,l+k)$ and that proves that $(z,l+k)$ is in the connective component of (x,g). $\square$

4.3.2. Action of morphisms

Let $\Phi : G\to G'$ be a morphism sending the group G to another group G'. We can define the action of Φ on $P\times G$ by the map

$$\varphi : P\times G\to P\times G'$$

$$\varphi(x,g)=(x,\Phi(g))$$

and then also its action on the graphs of set of vertices $V=P\times G$.

Property 4.4. *Let $\Phi : G \to G'$ be a surjective morphism. By denoting $\varphi : P \times G \to P \times G'$ the map defined by $\varphi(x,g) = (x, \Phi(g))$, the image $\varphi(\Gamma)$ of a periodic graph $\Gamma = ((P \times G), E)$ by φ is a periodic graph of period P and of group $G' = \Phi(G)$.*

In other words, the periodicity of the graphs is preserved by the surjective morphisms.

Proof. We have to prove that the graph $\varphi(\Gamma) = (P \times G', (\varphi \otimes \varphi)(E))$ is invariant by the translations $translation_{k'}$ acting on $P \times G'$ (where k' is any element of G'). It means that $((x,g'),(y,h')) \in (\varphi \otimes \varphi)(E)$ if and only if $((x,g'+k'),(y,h'+k')) \in (\varphi \otimes \varphi)(E)$.

From the surjectivity of Φ, we can assume that $k' = \Phi(k)$.

If $((x,g'),(y,h')) \in (\varphi \otimes \varphi)(E)$, then there exists $((x,g),(y,h)) \in E$ with $\Phi(g) = g'$ and $\Phi(h) = h'$. From the periodicity of Γ, we have $((x,g+k),(y,h+k)) \in E$ and it follows that $((x,\Phi(g)+\Phi(k)),(y,\Phi(h)+\Phi(k))) \in (\varphi \otimes \varphi)(E)$ namely that $((x,g'+k'),(y,h'+k')) \in (\varphi \otimes \varphi)(E)$. □

Notation and remark. If H is a subgroup of G having the property that the right and left cosets are equal, then G/H has a group structure and the projection

$$Projection_H : G \to G/H$$

which send each element $g \in G$ on its coset modulo H is a surjective morphism. It follows from Property 4.4 that the projection of any periodic graph by the map induced by such a morphism is periodic: the image of any periodic graph $\Gamma = (P \times G, E)$ by the map

$$projection_H : P \times G \to P \times G/H$$

$$projection_H(x,g) = (x, g \text{ } modulo \text{ } H)$$

is a periodic graph of group G/H and of period P.

4.3.3. Morphisms and connectivity

In the general case, the image of a connective component is included in the connective component of the image points. In the case of a periodic graph with a map induced by a morphism, this inclusion becomes an equality.

Property 4.5. *Let $\Phi : G \to G'$ be a morphism of groups. By denoting $\varphi : P \times G \to P \times G'$ the map defined by $\varphi(x,g) = (x, \Phi(g))$, the image of a connective component* component$_\Gamma(x,g)$ *of a periodic graph Γ by φ is equal to the connective component of the image point $\varphi(x,g) = (x, \Phi(g))$ in the graph $\varphi(\Gamma)$:*

$$\varphi(\text{component}_\Gamma(x,g)) = \text{component}_{\varphi(\Gamma)}\varphi(x,g).$$

Proof. In order to raise up the paths (it is not possible in the general case), we are going to prove that if $\varphi(\Gamma)$ has an edge between (x,g') and (y,h'), then every point of $\varphi^{-1}(x,g')$ is joined by an edge of Γ with a point of $\varphi^{-1}(y,h')$.

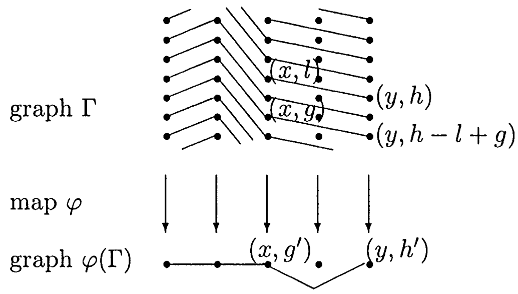

Fig. 6. The raising of an edge in the graph $f(\Gamma)$ in an edge coming from (x, l).

We assume that $\varphi(x, g) = (x, g')$ and that the image graph $\varphi(\Gamma)$ has an edge between (x, g') and (y, h'). By hypothesis, there exist two vertices (x, l) and (y, h) of respective images (x, g') and (y, h') which are related by an edge in Γ (Fig. 6).

By translating them from $-l + g$ with the property of periodicity, we obtain that (x, g) is related by an edge to $(y, h-l+g)$ whose image by φ is $(y, \Phi(h)-\Phi(l)+\Phi(g)) = (y, h' - g' + g') = (y, h')$. $\square$

4.3.4. The connective components invariant by a translation

We assume now that G is an abelian group and thus that the left and the right cosets from any subgroup H of G are equal. We consider a connective component $component_\Gamma(x, g)$ of a periodic graph $\Gamma = (P \times G, E)$ containing two elements of the form (y, h) and $(y, h + k)$. According to property 4.3, it is invariant by translation of element $k \in G$ and it leads to introduce the subgroup H of G generated by this element. The invariance of the connective component $component_\Gamma(x, g)$ by the translation of element k implies that it is the converse image of its image by $projection_H$ (we have denoted $Projection_H : G \to G/H$ the map defined by $Projection_H(g) = g$ *modulo* H and $projection_H : P \times G \to P \times (G/H)$ the map defined by $projection_H(x, g) = (x, Projection_H(g))$). We can write it

$$projection_H^{-1}(projection_H(component_\Gamma(x, g))) = component_\Gamma(x, g).$$

The projection $Projection_H : G \to G/H$ is a morphism and according to Property 4.5, we have

$$projection_H^{-1}(component_{projection_H(\Gamma)}(x, g \text{ modulo } H))) = component_\Gamma(x, g).$$

It means that the map $projection_H$ can be used in order to reduce the investigation of the connective component $component_\Gamma(x, g)$ to the one of $component_{projection_H(\Gamma)}(x, g \text{ modulo } H))$. This reduction is important because it allows in some cases to reduce the investigation of infinite connective components to the one of finite components.

4.4. Algorithm

We assume now that $\Gamma = (P \times G, E)$ is a periodic graph of finite period and of finitely generated abelian group because these assumptions provide a way to reduce the computation of the connective component of any vertex (x, g).

4.4.1. Property of infinite connective components

Lemma 4.2. *If a connective component* $\mathrm{component}_\Gamma(x, g)$ *has an infinite number of vertices, then it contains a pair of vertices of the form* (y, h) *and* $(y, h + k)$ *with an element* $k \in G$ *of infinite order.*

Proof. The map $\alpha : component_\Gamma(x, g) \to P$ defined by $\alpha(y, h) = y$ is not injective because the set $component_\Gamma(x, g)$ is infinite whereas its image is finite. We can even say that there exists an image $y \in P$ with an infinite converse image $\alpha^{-1}(y)$. There exists only a finite number of elements of finite order (i.e verifying $n.x = 0$ for $n \in \mathbb{N}^*$) because the torsion subgroup of any finitely generated abelian group is finite [7]. It follows that by denoting (y, h) as any element of $\alpha^{-1}(y)$, there exists $(y, h + k) \in \alpha^{-1}(y)$ with an element k of infinite order. $\square$

This lemma proves that the infinite connective components are invariant by some translations and by using the reasoning described in 4.3.4, it provides a way to reduce their investigation.

4.4.2. Algorithm

This algorithm computes the connective components in a periodic graph of finite period, of finitely generated abelian group and whose vertices have each a finite number of neighbours. It always finishes in a finite time on the contrary to the classical algorithm which only label the connected elements: it is deterministic.

Let (x, g) be a vertex of the periodic graph $\Gamma = (P \times G, E)$. In order to compute its connective component $component_\Gamma(x, g)$, we can use the following recursive process:

- If the group G is finite (its rank is null), the number of vertices $card(P \times G)$ is finite and we use the classical algorithm of labelling.
- If the group G is not finite, we follow the connective component of (x, g) by labelling its vertices (as in the classical algorithm).
  - If the connective component is finite, then the process ends in a finite time.
  - If the connective component is not finite, then we meet in a finite time two elements of the form (y, h) and $(y, h + k)$ with an element $k \in G$ of infinite order (there exists only a finite number of elements of G of finite order [7]). We denote H as the group generated by the element k. The investigation of the connective component $component_\Gamma(x, g)$ can be reduced to the one of $component_{projection_H(\Gamma)}(x, g \text{ } modulo \text{ } H)$ in the periodic graph $projection_H(\Gamma)$. Then we compute the new graph $projection_H(\Gamma)$: there is an edge between $(z, l) \in P \times (G/H)$ and $(z', l') \in P \times (G/H)$ if and only if there exist two neighbours in Γ, one in $projection_H^{-1}(z, l)$

and the other in $projection_H^{-1}(z', l')$. We do not have to investigate all the elements of $projection_H^{-1}(z, l)$ because by periodicity, there is an edge between a vertex of $projection_H^{-1}(z, l)$ and a vertex of $projection_H^{-1}(z', l')$ if and only if any vertex of $projection_H^{-1}(z, l)$ has a neighbour in $projection_H^{-1}(z', l')$. Then we just have to choose a vertex in $projection_H^{-1}(z, l)$ and to see if one of its neighbours (they are in finite number) is in $projection_H^{-1}(z', l')$.
We obtain the new graph $projection_H(\Gamma)$. Its properties are the same as Γ, and in order to investigate the connective component $component_{projection_H(\Gamma)}(x, g\ modulo\ H)$, we call recursively the process. The rank of the group G/H is the rank of G minus 1.

This process always ends because the rank of the finitely generated group decreases at each step.

5. Conclusion

Given any neighbourhood N of the origin in $\mathbb{Z}^n$, we have seen that the graph of N-adjacency of any part $Q \in \mathbb{Z}^n$ can be seen as a periodic graph of group G_Q (the stabilizer of Q) and of period P_Q (Property 3.1). This point of view is valid when the period is finite because several properties have allowed us to build a deterministic algorithm computing the connective component of a vertex in a periodic graph of finite period, of finitely generated group and whose vertices have a finite number of neighbours. It shows that the discrete connectivity of the subsets of $\mathbb{Z}^n$ of finite period can be decided in a finite time. It is the case of the rational digital hyperplanes (Theorem 2.2) and it provides a deterministic algorithm computing their connectivity. The situation is different for the irrational digital hyperplanes because their period is never finite (Theorem 2.2). Is their discrete connectivity decidable? The question remains open and could be the origin of deep ideas on these kind of questions.

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